$$u_{(9)} = u^{(4)} = u^*, \quad v_{(9)} = v^{(4)} = +1 = v^\circ, \quad D_{(9)}(w) \in W_0(w)$$

are valid. The most difficult part of the proof of assertion 8.9 is the proof of the following property of the pair $[u \neq u^{(4)},\ v = v_{(10)}]$:

$$T^1_{(7)}(g_1) < T^1_{(7)}(g_2)$$

where g_1 is the point where the trajectory goes onto the common boundary $G_{(9)}$ of regions $D_{(9)}$ and $D_{(10)}$, while g_2 is the point where the trajectory returns to this boundary.

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NECESSARY OPTIMALITY CONDITIONS IN A LINEAR PURSUIT PROBLEM

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Necessary conditions are presented for the optimality of a certain guaranteed time (upper layer time [1]) for a large class of pursuit problems. Sufficient conditions of a general form have been cited in [1-5] and in a number of other papers for the possibility of terminating the pursuit at a specified time and the guarantee time effectively computed. Sufficient optimality conditions for guarantee times have been discussed in [6-8].

1. Suppose that a linear pursuit problem in an n-dimensional Euclidean space R is described:

a) by linear vector differential equations

$$\dot{z} = Cz - u + v \tag{1}$$

where C is a constant nth-order square matrix, $u = u(t) \in P$ and $v = v(t) \in Q$ are vector-valued functions, measurable for $t \geqslant 0$, called the controls of the players (the pursuer and pursued respectively); $P \subset R$ and $Q \subset R$ are convex compacta;

b) by a terminal set M representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, and W_0 is some compact convex set in a subspace L

which is the orthogonal complement of M_0 in R.

We denote the orthogonal projection operator onto L by π, the dimension L by ν, and the unit sphere in L by K. We assume that $\nu \geqslant 2$. The aim of the pursuer is to bring the point z onto the set M, while the pursued player seeks to prevent this. We say that the pursuit can be terminated in a time $t\,(z_0)$ from the point z_0 if for an arbitrary control $v\,(t)$ of the pursued player, the pursuer can construct his own control $u\,(t)$ so that the point z hits onto the set M in a time not exceeding $t\,(z_0)$; the values of $z\,(s)$, $v\,(s)$ $(t - \varepsilon \leqslant s \leqslant t,\ \varepsilon > 0)$ are used for finding the value of parameter $u\,(t)$ at each instant t.

2. Consider the mapping h: $K \to L$ of the sphere K into space L, possessing the following properties:

a) the mapping h is a smooth homeomorphism,

b) every vector $\varphi \in K$ is normal to the surface $H = h\,(K)$ at the point $h\,(\varphi)$.

Let φ_0 be an arbitrary point of sphere K and let $s = (s^2, \ldots, s^\nu)$ be a local coordinate system in its neighborhood with origin 0 at the point φ_0, so that $\varphi = \varphi\,(s) = \varphi\,(s^2, \ldots, s^\nu)$. By $\varphi_j\,(s)$ we denote the vectors $\varphi_j\,(s) = \partial\varphi\,(s) / \partial s\ (j = 2, \ldots, \nu)$.

Definition. The surface $H = h\,(K)$, corresponding to the mapping h of sphere K into L, is said to be locally convex if h possesses properties (a) and (b) and, furthermore, if at each point $\varphi_0 \in K$ there is a positive-definite quadratic form with coefficients

$$h_{ij}\,(\varphi_0) = \left(\varphi_i\,(0) \cdot \frac{\partial h\,(\varphi\,(0))}{\partial s^j}\right) (i, j = 2, \ldots, \nu)$$

Lemma 1. Let the surface $H = h\,(K)$, corresponding to the mapping h of sphere K into L, be locally convex. Then there exist constants $C_1 < +\infty$ and $C_2 > 0$ such that the inequalities

$$(\varphi \cdot [(h\,(\varphi) - h\,(\psi)]) \leqslant C_1\,(\varphi \cdot [\varphi - \psi])$$
$$(\varphi \cdot [h\,(\varphi) - h\,(\psi)]) \geqslant C_2\,(\varphi \cdot [(\varphi - \psi]) \geqslant 0$$

are fulfilled for all $\varphi, \psi \in K$.

We do not prove here, for lack of space, Lemma 1 as well as Lemma 2. We remark that from inequalities (2) it follows, in particular, that the surface $H = h\,(K)$ is the boundary of some convex body W in L with a support function $(\varphi \cdot h\,(\varphi))$, so that for a point $x \in L$ to belong to W it is necessary and sufficient that the inequality $(\varphi \cdot [h\,(\varphi) - x]) \geqslant 0$ hold for all $\varphi \in K$

3. We assume that the following conditions have been fulfilled for problem (1).

Condition 1. For any $r > 0$ and any vector $\varphi \in K$ there exist unique vectors $u\,(r, \varphi) \in P$ and $v\,(r, \varphi) \in Q$ yielding the maximum of the following scalar products:

$$(\varphi \cdot e^{rC} u), \quad u \in P, \quad (\varphi \cdot e^{rC} v), \quad v \in Q$$

The surfaces $\pi e^{rC} u\,(r, K)$ and $\pi e^{rC} v\,(r, K)$ are locally convex; the mappings $u\,(r, \varphi)$ and $v\,(r, \varphi)$ are smooth mappings from $(0, +\infty) \times K$ into R.

Condition 2. For any $\varphi \in K$ there exists a unique vector $w_0\,(\varphi) \in W_0$ yielding the maximum of the expression

$$(\varphi \cdot w_0), \qquad w_0 \in W_0$$

and either the surface $\Sigma^\circ = w_0(K)$ is locally convex or the set W_0 consists of the single point 0. In the latter case we set $w_0(\varphi) \equiv 0,\ \varphi \in K$.

Suppose that Conditions 1 and 2 have been fulfilled for problem (1). Let t be an arbitrary nonnegative number. We construct a mapping of sphere K into L

$$W(t, \varphi) = w_0(\varphi) + \int_0^t \pi e^{rC}[u(r, \varphi) - v(r, \varphi)]\, dr \tag{3}$$

For an arbitrary positive t this mapping is, generally speaking, neither one-to-one nor regular. By $\Sigma^t = W(t, K)$ we denote the image of sphere K under mapping (3). It is easy to see that the vector φ is the normal to surface Σ^t at the point $W(t, \varphi)$. We assume the fulfillment of the following

Condition 3. The surface Σ^t is locally convex for each $t > 0$.

Lemma 2. Suppose that Conditions 1 – 3 have been fulfilled for problem (1). Then there exist continuous positive functions $\delta(t) \leqslant t$ and $c(t)$, defined on the interval $(0, +\infty)$, such that the inequality

$$\left(\psi \cdot \left\{\left[W(t, \psi) - \int_{t-\delta(t)}^{t} \pi e^{rC} u(r, \psi)\, dr\right] - \left[W(t, \varphi) - \int_{t-\delta(t)}^{t} \pi e^{rC} u(r, \varphi)\, dr\right]\right\}\right) \geqslant$$
$$\geqslant c(t)\,(\psi \cdot [\psi - \varphi])$$

is fulfilled for all $t > 0,\ \psi \in K,\ \varphi \in K$.

4. Let z be an arbitrary point of space R. The point, corresponding to it in L of the curve $\pi e^{tC} z$ can be, for some value t_0 of parameter t, captured by an "expanding" convex body $W(t)$ whose boundary is the locally convex surface $\Sigma^t = W(t, K)$. The function $W(t, \varphi)$ is continuous in $t,\ \varphi \in [0, +\infty) \times K$. Therefore, there exists a smallest nonnegative value of parameter t (let us call it $T(z)$) for which the inclusion

$$\pi e^{tC} z \in W(t) \tag{4}$$

holds. Obviously,

$$\pi e^{T(z)C} z \in \Sigma^{T(z)}$$

and consequently there exists a vector $\varphi(z) \in K$ such that

$$\pi e^{T(z)C} z = W(T(z), \varphi(z))$$

If, however, the point $\pi e^{tC} z$ lies outside the body $W(t)$ for any $t \geqslant 0$, we say that $T(z) = +\infty$.

Theorem 1. Suppose that Conditions 1 - 3 have been fulfilled for problem (1). Then, if the point $z_0 \in R$ is such that $0 < T(z_0) = T_0 < +\infty$, then the pursuit can be terminated in time T_0 from the point z_0

This theorem can be easily proved by the plan in [2] by reduction to Theorem 1 of [1]. However, its proof is subsumed in the proof of Theorem 2 following below.

5. For all $t \geqslant 0$ and $z \in R$ we define the continuous function (see [3])

$$\lambda(z, t) = \min_{\psi \in K} ([W(t, \psi) - \pi e^{tC} z] \cdot \psi)$$

In correspondence to what we said in Sect. 2, in order for inclusion (4) to hold it is necessary and sufficient to fulfill the inequality $\lambda(z, t) \geqslant 0$. Note that if $z \notin M$, then $\lambda(z, 0) < 0$, so that the number $T(z)$ is nothing else but the first positive root of the equation $\lambda(z, t) = 0$.

Theorem 2. Suppose that Conditions 1 - 3 have been fulfilled for problem (1). Let $z_0 \in R$ be such that $0 < T_0 = T(z_0) < +\infty$. Then in order for the time T_0 to be optimal, it is necessary that the inequality ($\varphi_0 = \varphi(z_0)$)

$$I(t, \tau) \equiv \lambda\left(e^{tC}\left(z_0 - \int_0^t e^{-rC}[u(T_0 - r, \varphi_0) - v(T_0 - r, \varphi_0)]\, dr\right), \tau\right) \leqslant 0$$

be fulfilled for all $t \in (0, T_0)$ and $\tau \in (0, T_0 - t)$.

We carry out the proof by contradiction. Suppose that the time T_0 is optimal and that $t_0 \in (0, T_0)$ and $\tau_0 \in (0, T_0 - t_0)$ are such that

$$I(t_0, \tau_0) = \lambda_0 > 0 \tag{5}$$

We set

$$\delta_0 = \min \delta(t), \qquad c_0 = \min c(t), \qquad t \in [\tau_0, T_0]$$

($\delta(t)$ and $c(t)$ are the functions given by Lemma 2) and we choose $\Delta > 0$ and a positive integer N such that $\Delta = t_0/N < \delta_0$. We assume that the pursuer constructs a sequence $\varepsilon_0 = 0 < \varepsilon_1 < \varepsilon_2 < \ldots$ of instants of choosing the control and inductively determines his own control on each of the semi-intervals $[0, \varepsilon_1)$, $[\varepsilon_1, \varepsilon_2)$, ... in the following manner. At the initial instant $t = 0$ the pursuer chooses $\varepsilon_1 = \Delta$ and on the semi-interval $[0, \varepsilon_1)$ sets his own control equal to $u(t) \equiv u(T_0 - t, \varphi_0)$. After this the pursued player, in the course of time, gives his own control $v(t)$ on $[0, \varepsilon_1)$. Moving in correspondence with these controls, the point $z(t)$ goes from the initial position z_0 to some position $z(\varepsilon_1)$.

Now suppose that both the pursuer and the pursued player have constructed their own controls on each of the semi-intervals $[0, \varepsilon_1)$, ..., $[\varepsilon_{k-1}, \varepsilon_k)$ ($k \geqslant 1$) and let $z(t)$ be the motion of point z corresponding to these controls. Then, the pursuer chooses ε_{k+1} from the following considerations: $\varepsilon_{k+1} = \varepsilon_k + \delta(T[z(\varepsilon_k)])$, if $k \geqslant N$ (or if $k < N$ but $T(z(\varepsilon_k)) < \tau_0$); $\varepsilon_{k+1} = \varepsilon_k + \Delta$ if $k < N$ and $T(z(\varepsilon_k)) \geqslant \tau_0$. Having chosen ε_{k+1}, he sets his own control on the semi-interval $[\varepsilon_k, \varepsilon_{k+1})$ equal to $u(t) \equiv u(T[z(\varepsilon_k)] - (t - \varepsilon_k), \varphi[z(\varepsilon_k)])$. After this the pursued player chooses his own control $v(t)$ on this same semi-interval, and the point z goes to a new position $z(\varepsilon_{k+1})$. In correspondence with the given inductive prescription for choosing the pursuer's control, each of the pursued player's control $v(t)$, $0 \leqslant t \leqslant T_0$ uniquely determines the corresponding motion $z(t)$ $0 \leqslant t \leqslant T_0$ ($z(0) = z_0$) of point z. It turns out that whatever be the pursued player's control $v(t)$, $0 \leqslant t \leqslant T_0$ the following alternative holds for $z(t)$: for any positive integer $k \geqslant 1$, either $T(z(\varepsilon_k)) = 0$ i.e., $z(\varepsilon_k) \in M$) or

$$0 < T(z(\varepsilon_k)) \leqslant T(z(\varepsilon_{k-1})) - (\varepsilon_k - \varepsilon_{k-1}) < +\infty \tag{6}$$

whence it follows immediately (see [3]) that from the point z_0 the pursuit can be terminated in a time no later than $T_0 = T(z_0)$.

Let us prove the alternative for $k = 1$. It is identical for $k > 1$. We have

$$z(\varepsilon_1) = z(\Delta) = e^{\Delta C}\left(z_0 - \int_0^{\Delta} e^{-rC}[u(T_0 - r, \varphi_0) - v(r)]\,dr\right).$$

Therefore, for any $\psi \in K$ we obtain, after simple manipulations,

$$(\psi\cdot[W(T_0-\Delta,\psi) - \pi e^{(T_0-\Delta)C} z(\Delta)]) = \left(\psi\cdot\left\{\left[W(T_0,\psi) - \int_{T_0-\delta(T_0)}^{T_0} \pi e^{rC} u(r,\psi)\,dr\right] - \right.\right.$$
$$\left.\left. - \left[W(T_0,\varphi_0) - \int_{T_0-\delta(T_0)}^{T_0} \pi e^{rC} u(r,\varphi_0)\,dr\right]\right\}\right) + \left(\psi\cdot \int_{T_0-\delta(T_0)}^{T_0-\Delta} \pi e^{rC}[u(r,\psi) - \right.$$
$$\left. - u(r,\varphi_0)]\,dr\right) + \left(\psi\cdot\int_{T_0-\Delta}^{T_0} \pi e^{rC}[v(r,\psi) - v(T_0 - r)]\,dr\right) \geqslant 0$$

Here the first term is nonnegative by virtue of Lemma 2, the second, by virtue of Condition 1 and of inequality (2), and the third, by virtue of the definition of $v(r,\psi)$. Thus

$$\pi e^{(T_0-\Delta)C} z(\Delta) \in W(T_0 - \Delta)$$

and, consequently, $T(z(\varepsilon_1)) \leqslant T_0 - \varepsilon_1$, q. e. d.

The time T_0 is optimal. Therefore, we can find a sequence of controls $v_i(s)$, $0 \leqslant s \leqslant T_0$ such that for the trajectories $z_i(s)$, $0 \leqslant s \leqslant T_0$ $(z_i(0) = z_0)$ corresponding to them (in the above-mentioned sense), the point $z_i(s)$ does not belong to M for all

$$s \in \left[0, T_0 - \frac{\Delta_0}{i}\right] \qquad \left(i = 1, 2, \ldots; \Delta_0 = \min\left\{\Delta, \frac{T_0 - t_0 - \tau_0}{2}\right\}\right)$$

The inequality

$$T(Z_i(\varepsilon_{ik})) \geqslant T_0 - \varepsilon_{ik} - \Delta_0/i \quad (0 \leqslant k \leqslant N, i = 1, 2, \ldots) \tag{7}$$

is obviously fulfilled for the trajectories $z_i(s)$, where ε_{ik} are the instants, determined by the $v_i(s)$ $(0 \leqslant s \leqslant T_0)$, of choosing the pursuer's control (otherwise, in correspondence with Theorem 1, a time no greater than $T(z_i(\varepsilon_{ik}))$ is left upto the end of the pursuit from point $z_i(\varepsilon_{ik})$ and, consequently, the time $T(z_i(\varepsilon_{ik})) + \varepsilon_{ik} < T_0 - \Delta_0/i$ is taken for the whole pursuit, which contradicts the definition of $z_i(s)$). From inequality (7), by an induction on k, it follows easily that

$$\varepsilon_{ik} \equiv k\Delta \qquad (k = 1, \ldots, N, \quad i = 1, 2, \ldots) \tag{8}$$

We introduce the notation

$$T_k = T_0 - k\Delta, \; z_i(k\Delta) = z_{ik}, \qquad T(z_{ik}) = T_{ik}, \; \varphi(z_{ik}) = \varphi_{ik}$$
$$(0 \leqslant k \leqslant N, \; i = 1, 2, \ldots)$$

From inequalities (6)—(8) follows

$$\lim_{i\to\infty} T_{ik} = T_k \qquad (k=0,1,\ldots,N) \tag{9}$$

Let us prove that the relations

$$\lim_{i\to\infty} \varphi_{ik} = \varphi_0 \qquad (k=0,1,\ldots,N) \tag{10}$$

$$\lim_{i\to\infty} z_{ik} = e^{k\Delta C}\left(z_0 - \int_0^{k\Delta} e^{-rC}\,[u\,(T_0 - r, \varphi_0) - v\,(T_0 - r, \varphi_0)]\,dr\right) \; (k=0,\ldots,N) \tag{11}$$

also hold

We prove equalities (10), (11) by induction on k. For $k = 0$ they are trivial because $\varphi_{i0} \equiv \varphi_0$, $z_{i0} \equiv z_0$. Suppose that relations (10), (11) are valid for some $k \leqslant N - 1$. We prove they also hold for $k + 1$. By virtue of the Cauchy formula

$$z_{ik+1} = e^{\Delta C}\left(z_{ik} - \int_0^{\Delta} e^{-sC}\,[u\,(T_{ik} - s, \varphi_{ik}) - v_i\,(k\Delta + s)]\,ds\right) \tag{12}$$

for any $\psi \in K$ we have

$$(\psi\cdot[W\,(T_{ik} - \Delta, \psi) - \pi e^{(T_{ik}-\Delta)C} z_{ik+1}]) = \left(\psi\cdot\left\{\left[W\,(T_{ik}, \psi) - \int_{T_{ik}-\Delta}^{T_{ik}} \pi e^{rC} u\,(r, \psi)\,dr\right] -\right.\right.$$

$$\left.\left. -\left[W\,(T_{ik}, \varphi_{ik}) - \int_{T_{ik}-\Delta}^{T_{ik}} \pi e^{rC} u\,(r, \varphi_{ik})\,dr\right]\right\}\right) + \left(\psi\cdot \int_{T_{ik}-\Delta}^{T_{ik}} \pi e^{rC}[v(r,\psi) - v_i(T_{ik} + k\Delta - r)]dr\right) \tag{13}$$

Hence, by virtue of Lemma 2, Condition 1, and the definition of $v(r, \psi)$, we obtain

$$(\psi\cdot\,[W\,(T_{ik} - \Delta, \psi) - \pi e^{(T_{ik}-\Delta)C} z_{ik+1}]) \geqslant c_0\,(\psi\cdot\,[\psi - \varphi_{ik}]) \tag{14}$$

Let us now assume that equality (10) is not fulfilled for $k + 1$, i.e., that there exists a subsequence $\{i_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \varphi_{i_n k+1} = \varphi^* \neq \varphi_0 \tag{15}$$

Then, by going to the limit in the equality

$$\pi e^{T_{i_n k+1} C} z_{i_n k+1} = W\,(T_{i_n k+1}, \varphi_{i_n k+1}) \tag{16}$$

and by using the continuity of $W\,(t, \varphi)$ and the formulas (9), (15), we obtain

$$\lim_{n\to\infty} \pi e^{T_{i_n k+1} C} z_{i_n k+1} = W\,(T_{k+1}, \varphi^*)$$

On the other hand (by virtue of the uniform boundedness of $|\,z_{ik}\,|$ with respect to i, k and of equality (9)), since

$$\lim_{n\to\infty} (\pi e^{T_{i_n k+1} C} z_{i_n k+1} - \pi e^{(T_{i_n k}-\Delta)C} z_{i_n k+1}) = 0 \tag{17}$$

by passing to the limit in inequality (14) with respect to the subsequence $\{i_n\}$ and by using the continuity of $W(t, \varphi)$ and formulas (9), (10) (for k), we obtain

$$(\psi[W(T_{k+1},\psi)-W(T_{k+1},\varphi^*)])\geqslant c_0(\psi\cdot[\psi-\varphi_0])$$

The latter is incorrect for $\psi=\varphi^*$. This contradiction proves equality (10).

Further, from relation (13), Lemma 2, and the definition of $v(r,\psi)$, for $\psi=\varphi_0$ we have

$$0\leqslant\left(\varphi_0\cdot\int\limits_{T_{ik}-\Delta}^{T_{ik}}\pi e^{rC}\,[v(r,\varphi_0)-v_i(T_{ik}+k\Delta-r)]\,dr\right)\leqslant$$
$$\leqslant(\varphi_0\cdot[W(T_{ik}-\Delta,\varphi_0)-\pi e^{(T_{ik}-\Delta)C}z_{ik+1}])$$

Going to the limit in formula (16) with $i_n\equiv n$ and using relations (9), (10), we obtain, keeping equality (17) in mind, that

$$\lim_{i\to\infty}\pi e^{(T_{ik}-\Delta)C}z_{ik+1}=W(T_{k+1},\varphi_0)$$

Hence

$$\lim_{i\to\infty}\left(\varphi_0\cdot\int\limits_{T_{ik}-\Delta}^{T_{ik}}e^{rC}\,[v(r,\varphi_0)-v_i(T_{ik}+k\Delta-r)]\,dr\right)=0 \tag{18}$$

By virtue of Filippov's theorem [10] there exists a measurable function $v^*(s)\in Q$, $T_{k+1}\leqslant\leqslant s\leqslant T_k$, such that

$$\lim_{i\to\infty}e^{T_{ik}C}\int\limits_0^{\Delta}e^{-sC}v_i(k\Delta+s)\,ds=\lim_{i\to\infty}\int\limits_{T_{ik}-\Delta}^{T_{ik}}e^{rC}v_i(T_{ik}+k\Delta-r)\,dr=\int\limits_{T_{k+1}}^{T_k}e^{rC}v^*(r)\,dr$$

From formula (18) and the definition of the function $v(r,\varphi_0)$ we then have that $v(r,\varphi_0)\equiv v^*(r)$ and

$$\lim_{i\to\infty}\int\limits_0^{\Delta}e^{-sC}v_i(k\Delta+s)\,ds=e^{-T_kC}\int\limits_{T_{k+1}}^{T_k}e^{rC}v(r,\varphi_0)\,dr=\int\limits_0^{\Delta}e^{-sC}v(T_0-k\Delta-s,\varphi_0)\,ds \tag{19}$$

The function $u(r,\varphi)$ is uniformly continuous on $[\tau_0,T_0]\times K$, therefore,

$$\lim_{i\to\infty}\int\limits_0^{\Delta}e^{-sC}u(T_{ik}-s,\varphi_{ik})\,ds=\int\limits_0^{\Delta}e^{-sC}u(T_0-k\Delta-s,\varphi_0)\,ds \tag{20}$$

Taking the relations (12), (9), (10), (11) (for k), (19), (20) into account, we obtain

$$\lim_{i\to\infty}z_{ik+1}=e^{\Delta C}\left(e^{k\Delta C}\left(z_0-\int\limits_0^{k\Delta}e^{-sC}\,[u(T_0-s,\varphi_0)-v(T_0-s,\varphi_0)]ds\right)\right)-$$
$$-e^{\Delta C}\int\limits_0^{\Delta}e^{-sC}u(T_0-k\Delta-s,\varphi_0)\,ds+e^{\Delta C}\int\limits_0^{\Delta}e^{-sC}v(T_0-k\Delta-s,\varphi_0)\,ds=$$
$$=e^{(k+1)\Delta C}\left(z_0-\int\limits_0^{(k+1)\Delta}e^{-sC}\,[u(T_0-s,\varphi_0)-v(T_0-s,\varphi_0)]\,ds\right)$$

Equality (11) is proved.

When $k = N$ equality (11) takes the form

$$\lim_{i\to\infty} z_i(t_0) = e^{t_0 C}\left(z_0 - \int_0^{t_0} e^{-sC}\left[u(T_0 - s, \varphi_0) - v(T_0 - s, \varphi_0)\right] ds\right)$$

Then, by virtue of the continuity of the function $\lambda(z, t)$ and of formula (5),

$$\lambda(z_i(t_0), \tau_0) \geqslant \lambda_0/2 > 0$$

for all sufficiently large i. Hence $T(z_i(t_0)) < \tau_0$. This contradicts formula (9). Theorem 2 is proved.

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